FUNCTIONS [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE Advanced

Single Correct Answer Type

1. Let R be the set of real numbers. If $f: R \to R$ is a function defined by $f(x) = x^2$, then f is
   a. injective but not surjective
   b. surjective but not injective
   c. bijective
   d. none of these **(IIT-JEE 1979)**

2. The entire graph of the equation $y = x^2 + kx - x + 9$ is strictly above the x-axis if and only if
   a. $k < 7$ b. $-5 < k < 7$
   c. $k > -5$ d. none of these
   (IIT-JEE 1979)

3. Let $f(x) = |x - 1|$. Then
   a. $f(x^2) = (f(x))^2$ b. $f(x + y) = f(x) + f(y)$
   c. $f(|x|) = |f(x)|$ d. none of these
   (IIT-JEE 1983)

4. If x satisfies $|x - 1| + |x - 2| + |x - 3| \geq 6$, then
   a. $0 \leq x \leq 4$ b. $x \leq -2$ or $x \geq 4$
   c. $x \leq 0$ or $x \geq 4$ d. none of these
   (IIT-JEE 1983)

5. If $f(x) = \cos(\log_e x)$, then $f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has value
   a. -1 b. $1/2$ c. -2 d. none of these
   (IIT-JEE 1983)

6. The domain of definition of the function
   $y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$ is
   a. $(-3, -2)$ excluding -2.5 b. $[0, 1]$ excluding 0.5
   c. $[-2, 1)$ excluding 0 d. none of these
   (IIT-JEE 1983)

7. Which of the following functions is periodic?
   a. $f(x) = x - [x]$, where $[x]$ denotes the largest integer less than or equal to the real number x
   b. $f(x) = \sin\frac{1}{x}$ for $x \neq 0$, $f(0) = 0$
   c. $f(x) = x\cos x$
   d. None of these **(IIT-JEE 1983)**

8. If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is
   a. $\left(\frac{1}{2}\right)^{x(x-1)}$ b. $\frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$
   c. $\frac{1}{2}\left(1 - \sqrt{1 + 4\log_2 x}\right)$ d. not defined
   (IIT-JEE 1992)

9. Let $f(x) = \sin x$ and $g(x) = \log_e|x|$. If the ranges of the composition functions fog and gof are R_1 and R_2, respectively, then
   a. $R_1 = \{u: -1 \leq u < 1\}$, $R_2 = \{v: -\infty < v < 0\}$
   b. $R_1 = \{u: -\infty < u < 0\}$, $R_2 = \{v: -\infty < v < 0\}$
   c. $R_1 = \{u: -1 < u < 1\}$, $R_2 = \{v: -\infty < v < 0\}$
   d. $R_1 = \{u: -1 \leq u \leq 1\}$, $R_2 = \{v: -\infty < v \leq 0\}$
   (IIT-JEE 1994)

10. Let $f(x) = (x + 1)^2 - 1$, $x \geq -1$. Then the set $\{x: f(x) = f^{-1}(x)\}$ is
    a. $\left\{0, -1, \frac{-3 + i\sqrt{3}}{2}, \frac{-3 - i\sqrt{3}}{2}\right\}$
    b. $\{0, 1, -1\}$
    c. $\{0, -1\}$
    d. empty **(IIT-JEE 1995)**

11. Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfies $f\left(\frac{x}{y}\right) = f(x) - f(y)$ for all x, y and $f(e) = 1$. Then
    a. $f(x)$ is bounded b. $f\left(\frac{1}{x}\right) \to 0$ as $x \to 0$
    c. $x f(x) \to 1$ as $x \to 0$ d. $f(x) = \log_e x$
    (IIT-JEE 1995)

12. The domain of definition of the function $f(x)$ given by the equation $2^x + 2^y = 2$ is
    a. $0 < x \leq 1$ b. $0 \leq x \leq 1$
    c. $-\infty < x \leq 0$ d. $-\infty < x < 1$
    (IIT-JEE 2000)

13. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$. Then for all x, $f(g(x))$ is equal to (where $[\cdot]$ represents the greatest integer function)
    a. x b. 1
    c. $f(x)$ d. $g(x)$ **(IIT-JEE 2001)**

14. If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals
    a. $\frac{\left(x + \sqrt{x^2 - 4}\right)}{2}$ b. $\frac{x}{1 + x^2}$
    c. $\frac{\left(x - \sqrt{x^2 - 4}\right)}{2}$ d. $1 + \sqrt{x^2 - 4}$
    (IIT-JEE 2001)

15. The domain of definition of $f(x)=\dfrac{\log_2(x+3)}{x^2+3x+2}$ is

a. $R-\{-1,-2\}$ b. $(-2,\infty)$
c. $R-\{-1,-2,-3\}$ d. $(-3,\infty)-\{-1,-2\}$

(IIT-JEE 2001)

16. Let $E=\{1, 2, 3, 4\}$ and $F=\{1, 2\}$. Then the number of onto functions from E to F is

a. 14 b. 16 c. 12 d. 8

(IIT-JEE 2001)

17. Let $f(x)=\dfrac{\alpha x}{(x+1)}, x \neq -1$. Then for what value of α is $f(f(x))=x$?

a. $\sqrt{2}$ b. $-\sqrt{2}$
c. 1 d. -1 **(IIT-JEE 2001)**

18. Suppose $f(x)=(x+1)^2$ for $x \geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y=x$, then $g(x)$ equals

a. $1-\sqrt{x}-1, x \geq 0$ b. $\dfrac{1}{(x+1)^2}, x>-1$
c. $\sqrt{x+1}, x \geq -1$ d. $\sqrt{x}-1, x \geq 0$

(IIT-JEE 2002)

19. Let the function $f: R \rightarrow R$ be defined by $f(x)=2x+\sin x$ for $x \in R$. Then f is

a. one-to-one and onto
b. one-to-one but not onto
c. onto but not one-to-one
d. neither one-to-one nor onto **(IIT-JEE 2002)**

20. If $f:[0,\infty) \rightarrow [0,\infty)$ and $f(x)=\dfrac{x}{1+x}$, then f is

a. one-one and onto b. one-one but not onto
c. onto but not one-one d. neither one-one nor onto

(IIT-JEE 2003)

21. The domain of definition of the function

$$f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$$

for real-valued x is

a. $\left[-\frac{1}{4}, \frac{1}{2}\right]$ b. $\left[-\frac{1}{2}, \frac{1}{2}\right]$
c. $\left(-\frac{1}{2}, \frac{1}{9}\right)$ d. $\left[-\frac{1}{4}, \frac{1}{4}\right]$

(IIT-JEE 2003)

22. The range of the function $f(x)=\dfrac{x^2+x+2}{x^2+x+1}, x \in R$, is

a. $(1,\infty)$ b. $(1, 11/7)$
c. $(1, 7/3]$ d. $(1, 7/5)$ **(IIT-JEE 2003)**

23. If $f(x)=\sin x+\cos x$, $g(x)=x^2-1$, then $g(f(x))$ is invertible in the domain

a. $\left[0, \frac{\pi}{2}\right]$ b. $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
c. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ d. $[0, \pi]$ **(IIT-JEE 2004)**

24. If the functions $f(x)$ and $g(x)$ are defined on $R \rightarrow R$ such that $f(x)=\begin{cases}0, & x \in \text{rational}\\ x, & x \in \text{irrational}\end{cases}$

and $g(x)=\begin{cases}0, & x \in \text{irrational}\\ x, & x \in \text{rational}\end{cases}$

then $(f-g)(x)$ is

a. one-one and onto b. neither one-one nor onto
c. one-one but not onto d. onto but not one-one

(IIT-JEE 2005)

25. X and Y are two sets and $f: X \rightarrow Y$. If $\{f(c)=y; c \subset X, y \subset Y\}$ and $\{f^{-1}(d)=x; d \subset Y, x \subset X\}$, then the true statement is

a. $f(f^{-1}(b))=b$ b. $f^{-1}(f(a))=a$
c. $f(f^{-1}(b))=b, b \subset y$ d. $f^{-1}(f(a))=a, a \subset x$

(IIT-JEE 2005)

26. Let $f(x)=x^2$ and $g(x)=\sin x$ for all $x \in R$. Then the set of all x satisfying $(fogogof)(x)=(gogof)(x)$, where $(fog)(x)=f(g(x))$, is

a. $\pm\sqrt{n\pi}, n \in \{0, 1, 2, \ldots\}$
b. $\pm\sqrt{n\pi}, n \in \{1, 2, \ldots\}$
c. $\frac{\pi}{2}+2n\pi, n \in \{\ldots, -2, -1, 0, 1, 2\ldots\}$
d. $2n\pi, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ **(IIT-JEE 2011)**

Multiple Correct Answers Type

1. If $y=f(x)=\dfrac{(x+2)}{(x-1)}$, then

a. $x=f(y)$
b. $f(1)=3$
c. y increases with x for $x<1$
d. f is a rational function of x **(IIT-JEE 1984)**

2. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at $(0, 0)$ and $(x, g(x))$ is $\sqrt{3}/4$, then the function $g(x)$ is

a. $g(x)=\pm\sqrt{1-x^2}$ b. $g(x)=\sqrt{1-x^2}$
c. $g(x)=-\sqrt{1-x^2}$ d. $g(x)=\sqrt{1+x^2}$

(IIT-JEE 1989)

3. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where $[x]$ stands for the greatest integer function, then

a. $f\left(\frac{\pi}{2}\right) = -1$ b. $f(\pi) = 1$

c. $f(-\pi) = 0$ d. $f\left(\frac{\pi}{4}\right) = 1$

(IIT-JEE 1991)

4. If $f(x) = 3x - 5$, then $f^{-1}(x)$

a. is given by $\frac{1}{(3x-5)}$

b. is given by $\frac{(x+5)}{3}$

c. does not exist because f is not one-one

d. does not exist because f is not onto

(IIT-JEE 1998)

5. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then

a. $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$

b. $f(x) = \sin x$, $g(x) = |x|$

c. $f(x) = x^2$, $g(x) = \sin\sqrt{x}$

d. f and g cannot be determined **(IIT-JEE 1998)**

6. Let $f: (-1, 1) \to R$ be such that $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$ for $\theta \in 0\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ is (are)

a. $1-\sqrt{\frac{3}{2}}$ b. $1+\sqrt{\frac{3}{2}}$ c. $1-\sqrt{\frac{2}{3}}$ d. $1+\sqrt{\frac{2}{3}}$

(IIT-JEE 2012)

7. Let $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then

a. $f(x)$ is an odd function b. $f(x)$ is a one-one function

c. $f(x)$ is an onto function d. $f(x)$ is an even function

(JEE Advanced 2014)

8. Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \frac{\pi}{2}\sin x$ for all $x \in R$. Let (fog) (x) denote $f(g(x))$ and (gof) (x) denote $g(f(x))$. Then which of the following is (are) true?

a. Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

b. Range of fog is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

c. $\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{\pi}{6}$

d. There is an $x \in R$ such that (gof) $(x) = 1$

(JEE Advanced 2015)

Matching Column Type

1. Let $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6}$.

Match the expressions/statements given in Column I with expressions/statements given in Column II.

Column I	Column II
(a) If $-1 < x < 1$, then $f(x)$ satisfies	(p) $0 < f(x) < 1$
(b) If $1 < x < 2$, then $f(x)$ satisfies	(q) $f(x) < 0$
(c) If $3 < x < 5$, then $f(x)$ satisfies	(r) $f(x) > 0$
(d) If $x > 5$, then $f(x)$ satisfies	(s) $f(x) < 1$

(IIT-JEE 2007)

2. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(a) The number of solutions of the equation $xe^{\sin x} - \cos x = 0$ in the interval $\left(0, \frac{\pi}{2}\right)$	(p) 1
(b) Value(s) of k for which the planes $kx + 4y + z = 0$, $4x + ky + 2z = 0$ and $2x + 2y + z = 0$ intersect in a straight line	(q) 2
(c) Value(s) of k for which $\lvert x-1\rvert + \lvert x-2\rvert + \lvert x+1\rvert + \lvert x+2\rvert = 4k$ has integer solution(s)	(r) 3
(d) If $y' = y + 1$ and $y(0) = 1$, then value(s) of $y(\log_e 2)$	(s) 4
	(t) 5

(IIT-JEE 2009)

3. Match the statements given in Column I with the intervals/union of intervals given in Column II.

Column I	Column II
(a) The set $\left\{\mathrm{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number}, \lvert z\rvert = 1, z \neq \pm 1\right\}$ is	(p) $(-\infty, -1) \cup (1, \infty)$
(b) The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty, 0) \cup (0, \infty)$

(c) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta): 0 \le \theta < \frac{\pi}{2}\right\}$ is	(r) $[2, \infty)$
(d) If $f(x) = x^{3/2}(3x - 10)$, $x \ge 0$, then $f(x)$ is increasing in	(s) $(-\infty, -1] \cup [1, \infty)$
	(t) $(-\infty, 0] \cup [2, \infty)$

(IIT-JEE 2011)

Fill in the Blanks Type

1. The values of $f(x) = 3\sin\left(\sqrt{\frac{\pi^2}{16} - x^2}\right)$ lie in the interval________ **(IIT-JEE 1983)**
2. The domain of the function $f(x) = \sin^{-1}\left\{\log_2 \frac{x^2}{2}\right\}$ is given by________ **(IIT-JEE 1984)**
3. Let A be a set of n distinct elements. Then the total number of distinct functions from A to A is ________ and out of these, _______ are onto functions. **(IIT-JEE 1985)**
4. If $f(x) = \sin \log_e \left\{\frac{\sqrt{4 - x^2}}{1 - x}\right\}$, then the domain of $f(x)$ is _____ and its range is _____. **(IIT-JEE 1985)**
5. There are exactly two distinct linear functions, _______ and _______, which map $[-1, 1]$ onto $[0, 2]$. **(IIT-JEE 1985)**
6. If f is an even function defined on the interval $(-5, 5)$, then four real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ are _______, _______, _______, and _______. **(IIT-JEE 1985)**
7. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then $(gof)(x) =$ _______. **(IIT-JEE 1996)**

True/False Type

1. If $f(x) = (a - x^n)^{1/n}$, where $a > 0$ and n is a positive integer, then $f[f(x)] = x$. **(IIT-JEE 1983)**
2. The function $f(x) = \frac{(x^2 + 4x + 30)}{(x^2 - 8x + 18)}$ is not onto. **(IIT-JEE 1983)**
3. If $f_1(x)$ and $f_2(x)$ are defined on the domain D_1 and D_2, respectively, then $f_1(x) + f_2(x)$ is defined on $D_1 \cup D_2$. **(IIT-JEE 1988)**

Subjective Type

1. Find the domain and range of the function $f(x) = \frac{x^2}{1 + x^2}$. Is the function one-to-one? **(IIT-JEE 1978)**
2. Draw the graph of $y = |x|^{1/2}$ for $-1 \le x \le 1$. **(IIT-JEE 1978)**
3. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, find $f(6)$. **(IIT-JEE 1979)**
4. Let f be a one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statements is true and the remaining two are false: $f(x) = 1$, $f(y) \ne 1$, and $f(z) \ne 2$. Determine $f^{-1}(1)$. **(IIT-JEE 1982)**
5. Find the natural number a for which $\sum_{k=1}^{n} f(a + k) = 16(2^n - 1)$, where the function f satisfies the relation $f(x + y) = f(x)\, f(y)$ for all natural numbers x, y and, further, $f(1) = 2$. **(IIT-JEE 1992)**
6. Let $\{x\}$ and $[x]$ denote the fractional and integral parts of a real number x, respectively. Solve $4\{x\} = x + [x]$. **(IIT-JEE 1992)**
7. A function $f: R \to R$ is defined by $f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the interval of values of α for which f is onto. Is the function one-to-one for $\alpha = 3$? Justify your answer. **(IIT-JEE 1996)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. d. **2.** b. **3.** d. **4.** c. **5.** d.
6. c. **7.** a. **8.** b. **9.** d. **10.** c.
11. d. **12.** d. **13.** b. **14.** a. **15.** d.
16. a. **17.** d. **18.** d. **19.** a. **20.** b.
21. a. **22.** c. **23.** b. **24.** a. **25.** d.
26. a.

Multiple Correct Answers Type

1. a., d. **2.** b., c. **3.** a., c. **4.** b.
5. a. **6.** a., b. **7.** a., b., c. **8.** a.,b.,c.

Matching Column Type

1. (a) – (p), (r), (s); (b) – (q), (s); (c) – (q), (s);
(d) – (p), (r), (s)

2. (c) – (q), (r), (s), (t)

3. (b) – (t)

Fill in the Blanks Type

1. $\left[0, \frac{3}{\sqrt{2}}\right]$ **2.** $[-2, -1] \cup [1, 2]$ **3.** n^n, $n!$

4. $(-2, 1)$, $[-1, 1]$ **5.** $x + 1$ and $-x + 1$

6. $\frac{-3 \pm \sqrt{5}}{2}, \frac{3 \pm \sqrt{5}}{2}$ **7.** 1

True/False Type

1. True **2.** True **3.** False

Subjective Type

1. R, $[0, 1)$; f is not one-to-one

3. 3 **4.** y **5.** $a = 3$

6. $0, \frac{5}{3}$ **7.** $2 \le \alpha \le 14$, No

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. d. $f(x) = x^2$ is many-one as $f(1) = f(-1) = 1$.
Also, f is into, as the range of function is $[0, \infty)$ which is subset of R (co-domain).
Therefore, f is neither injective nor surjective.

2. b. $y = x^2 + (k-1)x + 9 = \left(x + \frac{k-1}{2}\right)^2 + 9 - \left(\frac{k-1}{2}\right)^2$

For entire graph to be above x-axis, we should have

$$9 - \left(\frac{k-1}{2}\right)^2 > 0$$

or $k^2 - 2k - 35 < 0$ or $(k-7)(k+5) < 0$

or $-5 < k < 7$

3. d. $f(x) = |x - 1|$

$\therefore f(x^2) = |x^2 - 1|$ and $(f(x))^2 = |x-1|^2 = x^2 - 2x + 1$

Therefore $f(x^2) \neq (f(x))^2$

Hence, option (a) is not true.

$f(x + y) = f(x) + f(y)$ or $|x + y - 1| = |x - 1| + |y - 1|$, which is absurd. Put $x = 2$, $y = 3$ and verify.

Hence, option (b) is not true.

Consider $f(|x|) = |f(x)|$.

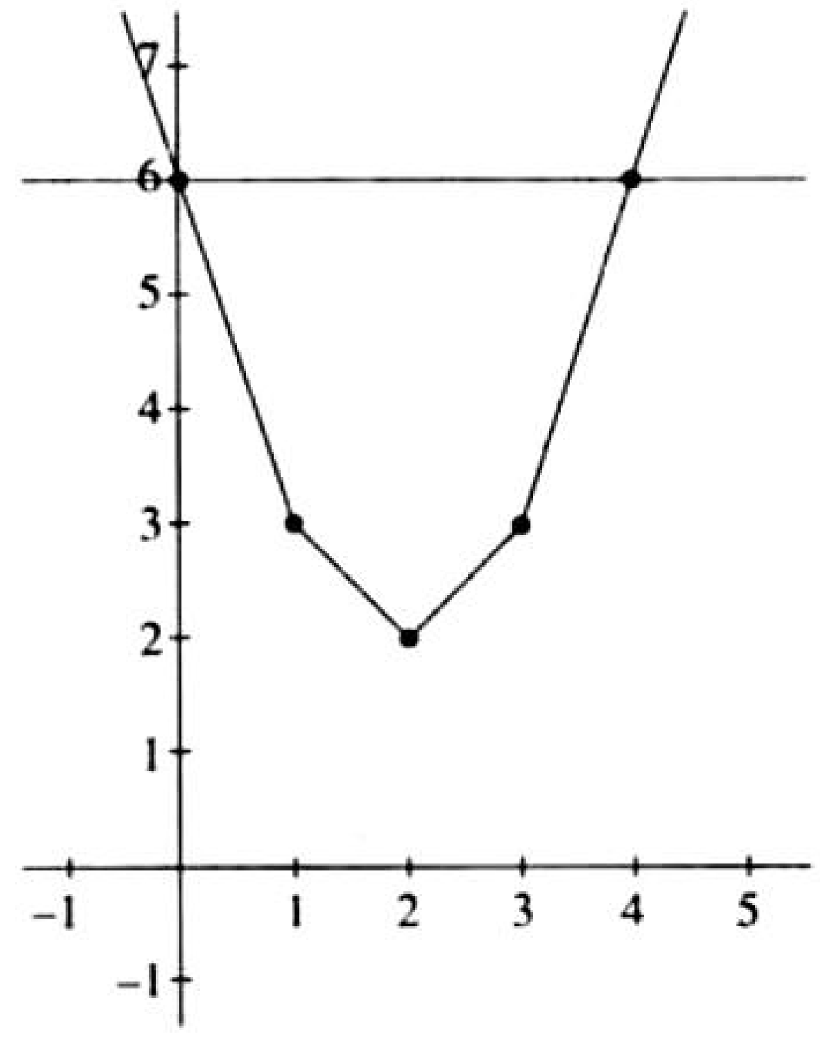
Put $x = -5$. Then $f(|-5|) = f(5) = 4$ and $|f(-5)| = |-5 - 1| = 6$.
Therefore, (c) is not correct.

4. c. We have to solve
$|x - 1| + |x - 2| + |x - 3| \geq 6$.
Let $|x - 1| + |x - 2| + |x - 3| < 6$
Now $|(x - 1) + (x - 2) + (x - 3)| < |x - 1| + |x - 2| + |x - 3| < 6$
$\therefore |3x - 6| < 6$
$\therefore |x - 2| < 2$
$\therefore -2 < x - 2 < 2$
or $0 < x < 4$
Hence, for $|x - 1| + |x - 2| + |x - 3| \geq 6$, $x \leq 0$ or $x \geq 4$.
Alternative Method:
We have $f(x) = |x - 1| + |x - 2| + |x - 3|$
To draw the graph of the function, we consider

$f(0) = 1 + 2 + 3 = 6$
$f(1) = 0 + 1 + 2 = 3$
$f(2) = 1 + 0 + 1 = 2$
$f(3) = 2 + 1 + 0 = 3$
$f(4) = 3 + 2 + 1 = 6$

Now, join these points with line segments as shown in the following figure.
Note: The methods to draw the graphs of such functions is explained in the book "Graphs for JEE Main and JEE Advanced".

Also, draw the line $y = 6$



For $f(x) = |x-1| + |x-2| + |x-3| \geq 6$, graph $y = f(x)$ must lie above the graph of $y = 6$.
As shown in the figure it occurs for $x \leq 0$ or $x \geq 4$.

5. d. $f(x) = \cos(\log x)$

$$\therefore f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$$

$$= \cos(\log x)\cos(\log y) - \frac{1}{2}[\cos(\log x - \log y) + \cos(\log x + \log y)]$$

$$= \cos(\log x)\cos(\log y) - \frac{1}{2}[2\cos(\log x)\cos(\log y)]$$

$$= 0$$

6. c. We have $y = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

or $\quad y = f(x) + g(x)$

Then, the domain of given function is $D_f \cap D_g$.

Now, for the domain of $f(x) = \dfrac{1}{\log_{10}(1-x)}$,

we know it is defined only when $1 - x > 0$ and $1 - x \neq 1$

or $x < 1$ and $x \neq 0$. Therefore, $D_f = (-\infty, 1) - \{0\}$.

For the domain of $g(x) = \sqrt{x+2}$,

$x + 2 \geq 0$ or $x \geq -2$

$\therefore D_g = [-2, \infty)$

Therefore, common domain is $[-2, 1) - \{0\}$.

7. a. $f(x) = \{x\}$ is periodic with period 1.

$$f(x) = \sin\frac{1}{x} \text{ for } x \neq 0; f(0) = 0$$

is non-periodic as $g(x) = \dfrac{1}{x}$ is non-periodic.

Also, $f(x) = x\cos x$ is non-periodic as $g(x) = x$ is non-periodic.

8. b. $y = 2^{x(x-1)}$

$\therefore y = 2^{x^2 - x}$

$\therefore x^2 - x = \log_2 y$

$\therefore x^2 - x - \log_2 y = 0$

$$\therefore x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\log_2 y}\right)$$

Since $x \in [1, \infty)$, we choose $x = \dfrac{1}{2}\left(1 + \sqrt{1 + 4\log_2 y}\right)$

or $f^{-1}(x) = \dfrac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$.

9. d. We have $fog(x) = f(g(x)) = \sin(\log_e |x|)$.

$\log_e|x|$ has range R, for which $\sin(\log_e|x|) \in [-1, 1]$.

Therefore, $R_1 = \{u: -1 \leq u \leq 1\}$.

Also, $gof(x) = g(f(x)) = \log_e |\sin x|$.

$\because 0 \leq |\sin x| \leq 1$

$-\infty < \log_e |\sin x| \leq 0$

or $R_2 = \{v: -\infty < v \leq 0\}$

10. c. Since $f(x) = (x+1)^2 - 1$ is continuous function, solution of $f(x) = f^{-1}(x)$ lies on the line $y = x$. Therefore,

$f(x) = f^{-1}(x) = x$

or $(x+1)^2 - 1 = x$

or $x^2 + x = 0$

i.e., $x = 0$ or -1

Therefore, the required set is $\{0, -1\}$.

11. d. $f(x)$ is continuous for all $x > 0$ and

$$f\left(\frac{x}{y}\right) = f(x) - f(y)$$

Also, $f(e) = 1$.

Therefore, clearly, $f(x) = \log_e x$ which satisfies all these properties.

Clearly, $f(x) = \log_e x$, is an unbounded function.

12. d. It is given that $2^x + 2^y = 2 \ \forall \ x, y \in R$

or $2^y = 2 - 2^x$

or $y = \log_2(2 - 2^x)$

Therefore, function is defined only when $2 - 2^x > 0$ or $2^x < 2$ or $x < 1$.

13. b. $g(x) = 1 + \{x\}, f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

where $\{x\}$ represents the fractional part function. Therefore,

$$f(g(x)) = \begin{cases} -1, & 1 + \{x\} < 0 \\ 0, & 1 + \{x\} = 0 \\ 1, & 1 + \{x\} > 0 \end{cases}$$

$$= 1,\ 1 + \{x\} > 0 \quad (\because 0 \leq \{x\} < 1)$$

$$= 1 \ \forall \ x \in R$$

14. a. $f: [1, \infty) \to [2, \infty)$

$$f(x) = x + \frac{1}{x} = y$$

or $\quad x^2 - yx + 1 = 0$

or $\quad x=\dfrac{y\pm\sqrt{y^2-4}}{2}$

But given $f:[1,\infty)\to[2,\infty)$

$\therefore \quad x=\dfrac{y+\sqrt{y^2-4}}{2}$

or $\quad f^{-1}(x)=\dfrac{x+\sqrt{x^2-4}}{2}$

15. d. For domain of $f(x)=\dfrac{\log_2(x+3)}{x^2+3x+2}$,

$x^2+3x+2\neq 0$ and $x+3>0$

or $x\neq -1,-2$ and $x>-3$

$\therefore D_f=(-3,\infty)-\{-1,-2\}$.

16. a. From E to F we can define, in all, $2\times2\times2\times2=16$ functions (2 options for each element of E) out of which 2 are into, when all the elements of E map to either 1 or 2. Therefore,
Number of onto functions = 16 – 2 = 14

17. d. $f(x)=\dfrac{\alpha x}{x+1}, x\neq -1$

$$f(f(x))=x \text{ or } \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1}+1}=x$$

or $\dfrac{\alpha^2 x}{(\alpha+1)x+1}=x$

or $(\alpha+1)x^2+(1-\alpha^2)x=0 \quad (1)$

or $\quad \alpha+1=0$ and $1-\alpha^2=0$

[As it is true $\forall\, x\neq -1$, Eq. (1) is an identity]

or $\quad \alpha=-1$

18. d. Given that $f(x)=(x+1)^2, x\geq -1$.
Now, if $g(x)$ is the reflection of $f(x)$ in the line $y=x$, then $g(x)$ is an inverse function of $y=f(x)$.
Given $y=(x+1)^2 \quad (x\geq -1$ and $y\geq 0)$

or $x=\pm\sqrt{y}-1$

or $g(x)=f^{-1}(x)=\sqrt{x}-1, x\geq 0$

19. a. Given that $f(x)=2x+\sin x, x\in R$

or $f'(x)=2+\cos x$

But $-1\leq\cos x\leq 1$

or $1\leq 2+\cos x\leq 3$

$\therefore f'(x)>0\ \forall\, x\in R$

Therefore, $f(x)$ is strictly increasing and, hence, one-one.

Also, as $x\to\infty, f(x)\to\infty$, and $x\to-\infty, f(x)\to-\infty$. Further $f(x)$ is continuous. Therefore,

Range of $f(x)=R=$ co-domain of $f(x)$

Hence, $f(x)$ is onto.

Thus, $f(x)$ is one-one and onto.

20. b. Given that $f:[0,\infty)\to[0,\infty), f(x)=\dfrac{x}{x+1}$

or $f'(x)=\dfrac{1+x-x}{(1+x)^2}=\dfrac{1}{(1+x)^2}>0\ \forall\, x$

Therefore, f is an increasing function or f is one-one.

Also, $D_f=[0,\infty)$.

And for range, let $\dfrac{x}{1+x}=y$ or $x=\dfrac{y}{1-y}$.

Since $x\geq 0$, $\dfrac{y}{1-y}\geq 0, \therefore \dfrac{y}{y-1}\leq 0 \therefore 0\leq y<1$. Therefore, $R_f=[0,1)\neq$ co-domain.

Thus, f is not onto.

21. a. For $f(x)=\sqrt{\sin^{-1}(2x)+\dfrac{\pi}{6}}$ to be defined and real,

$\sin^{-1}2x+\pi/6\geq 0$

or $\sin^{-1}2x\geq -\dfrac{\pi}{6} \quad (1)$

But we know that $-\pi/2\leq\sin^{-1}2x\leq\pi/2 \quad (2)$

Combining (1) and (2),

$-\pi/6\leq\sin^{-1}2x\leq\pi/2$

$\therefore \sin(-\pi/6)\leq 2x\leq\sin(\pi/2)$

$\therefore -1/2\leq 2x\leq 1$

$\therefore -1/4\leq x\leq 1/2$

$\therefore D_f=\left[-\dfrac{1}{4},\dfrac{1}{2}\right]$

22. c. We have $f(x)=\dfrac{x^2+x+2}{x^2+x+1}=\dfrac{(x^2+x+1)+1}{x^2+x+1}$

$$=1+\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

We can see here that as $x\to\infty, f(x)\to 1$.

Also, $f(x)$ is maximum when $\left(x+\dfrac{1}{2}\right)^2+\dfrac{3}{4}$ is minimum which is so when $x=-1/2$.

Therefore, $f_{\max}=1+\dfrac{1}{3/4}=\dfrac{7}{3}$.

$\therefore R_f=(1,7/3]$

Alternative Method:

We have $y=\dfrac{x^2+x+2}{x^2+x+1}$

$\therefore \quad yx^2+yx+y=x^2+x+2$

$\therefore \quad (y-1)x^2+(y-1)x+y-2=0$

Clearly, $y\neq 1$.

Since x is real,

$D\geq 0$

$\therefore \quad (y-1)^2-4(y-1)(y-2)\geq 0$

$\therefore \quad (y-1)[y-1-4y+8]\geq 0$

$\therefore \quad (y-1)(3y-7)\leq 0$

$\therefore \quad y\in(1,7/3]$

23. b. $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$

$\therefore g(f(x)) = (\sin x + \cos x)^2 - 1 = \sin 2x$

Clearly, $g(f(x))$ is invertible in

$-\frac{\pi}{2} \le 2x \le \frac{\pi}{2}$

($\because \sin\theta$ is invertible when $-\pi/2 \le \theta \le \pi/2$)

or $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$

24. a. We are given that

$$f: R \to R, f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}$$

$$g: R \to R, g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$$

$\therefore (f - g): R \to R$ such that

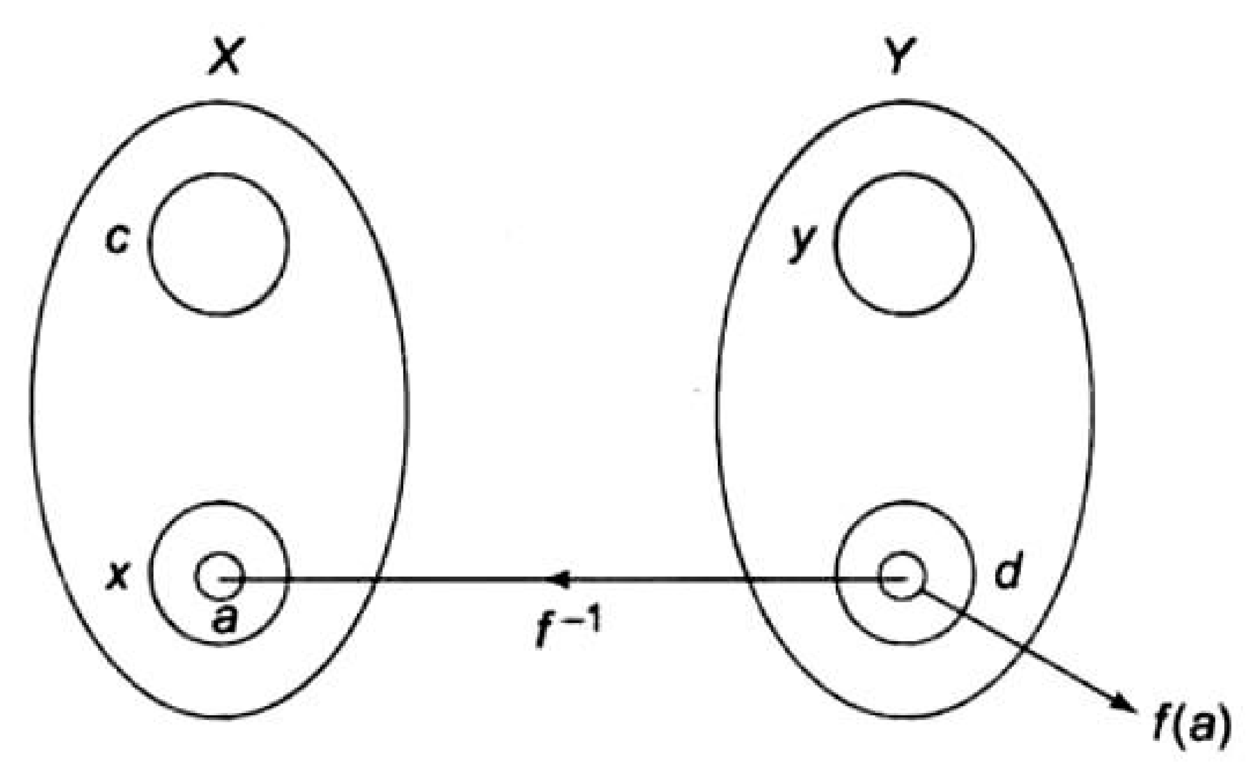
$$(f-g)(x) = \begin{cases} -x, & \text{if } x \in \text{rational} \\ x, & \text{if } x \in \text{irrational} \end{cases}$$

Since $(f - g)$: $R \to R$ for any x, there is only one value of $(f(x) - g(x))$ whether x is rational or irrational. Moreover, as $x \in R$, $f(x) - g(x)$ also belongs to R. Therefore, $(f - g)$ is one-one and onto.

25. d. Given that X and Y are two sets and $f: X \to Y$.

$\{f(c) = y; c \subset X, y \subset Y\}$ and $\{f^{-1} d = x; d \subset Y, x \subset X\}$.

The pictorial representation of given information is as shown in the figure.



Since $f^{-1} d = x$,

$f(x) = d$

Now, if $a \subset x \Rightarrow f(a) \subset f(x) = d$, then

$f^{-1}[fa] = a$

Therefore, $f^{-1}(f(a)) = a$; $a \subset x$ is the correct option.

26. a. $(fogogof)(x) = \sin^2(\sin x^2)$

$(gogof)(x) = \sin(\sin x^2)$

$\therefore \quad \sin^2(\sin x^2) = \sin(\sin x^2)$

$\Rightarrow \quad \sin(\sin x^2)[\sin(\sin x^2) - 1] = 0$

$\Rightarrow \quad \sin(\sin x^2) = 0$ or 1

$\Rightarrow \quad \sin x^2 = n\pi$ or $2m\pi + \pi/2$, where $m, n \in I$

$\Rightarrow \quad \sin x^2 = 0$

$\Rightarrow \quad x^2 = n\pi \Rightarrow x = \pm\sqrt{n\pi}$, $n \in \{0, 1, 2, \ldots\}$.

Multiple Correct Answers Type

1. a., d. Given that $f(x) = y = \frac{x+2}{x-1}$

a. Let $f(x) = \frac{x+2}{x-1} = y$ or $x + 2 = xy - y$

or $x = \frac{2+y}{y-1}$ or $x = f(y)$

Therefore, (a) is correct.

b. $f(1) \ne 3$. Therefore, (b) is not correct.

c. $f'(x) = \frac{x-1-x-2}{(x-1)^2} = \frac{-3}{(x-1)^2} < 0$ for $\forall x \in R - \{1\}$

Therefore, $f(x)$ is decreasing $\forall x \ne 1$.

Thus, (c) is not correct.

d. Clearly $f(x) = \frac{x+2}{x-1}$ is a rational function of x.

Thus, (d) is the correct answer.

2. b., c. As $(0, 0)$ and $(x, g(x))$ are two vertices of an equilateral triangle, length of the side of Δ is

$$\sqrt{(x-0)^2 + (g(x)-0)^2} = \sqrt{x^2 + (g(x))^2}$$

$\therefore$ Area of equilateral $(\Delta) = \frac{\sqrt{3}}{4}\left(x^2 + (g(x))^2\right)$

$= \frac{\sqrt{3}}{4}$ (given)

or $g(x)^2 = 1 - x^2$

or $g(x) = \pm\sqrt{1 - x^2}$

Therefore, (b) and (c) are the correct answers as (a) is not a function (since image of x is not unique).

3. a., c. $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

We know $9 < \pi^2 < 10$ and $-10 < -\pi^2 < -9$

$\therefore [\pi^2] = 9$ and $[-\pi^2] = -10$

$\therefore f(x) = \cos 9x + \cos(-10x)$

$= \cos 9x + \cos 10x$

a. $f\left(\frac{\pi}{2}\right) = \cos\frac{9\pi}{2} + \cos 5\pi = -1$ (true)

b. $f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ (false).

c. $f(-\pi) = \cos(-9\pi) + \cos(-10\pi) = \cos 9\pi + \cos 10\pi$

$= -1 + 1 = 0$ (true)

d. $f\left(\frac{\pi}{4}\right) = \cos\frac{9\pi}{4} + \cos\frac{5\pi}{2} = \cos\left(2\pi + \frac{\pi}{4}\right) + 0 = \frac{1}{\sqrt{2}}$ (false)

Thus, (a) and (c) are correct options.

4. b. $f(x) = 3x - 5$ (given)

Let $y = f(x) = 3x - 5$

or $y+5=3x$ or $x=\dfrac{y+5}{3}$

or $f^{-1}(x)=\dfrac{x+5}{3}$

5. a. If $f(x)=\sin^2 x$ and $g(x)=\sqrt{x}$

$$fog=f(g(x))=f(\sqrt{x})=\sin^2\sqrt{x}$$

and $gof(x)=g(f(x))=g(\sin^2 x)=\sqrt{\sin^2 x}=|\sin x|$

Again, if $f(x)=\sin x$, $g(x)=|x|$,

$fog(x)=f(g(x))=f(|x|)=\sin|x|\neq(\sin\sqrt{x})^2$

When $f(x)=x^2$, $g(x)=\sin\sqrt{x}$,

$fog(x)=f[g(x)]=f(\sin\sqrt{x})=(\sin\sqrt{x})^2$

and $(gof)(x)=g[f(x)]=g(x^2)$

$$=\sin\sqrt{x^2}=\sin|x|\neq|\sin x|$$

6. a., b. $f(\cos 4\theta)=\dfrac{2}{2-\sec^2\theta}$

For $\theta\in\left(0,\dfrac{\pi}{4}\right)\cup\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$

Let $\cos 4\theta=\dfrac{1}{3}$

$\Rightarrow\quad \cos 2\theta=\pm\sqrt{\dfrac{1+\cos 4\theta}{2}}=\pm\sqrt{\dfrac{2}{3}}$

$\therefore\quad f\left(\dfrac{1}{3}\right)=\dfrac{2}{2-\sec^2\theta}=\dfrac{2\cos^2\theta}{2\cos^2\theta-1}=1+\dfrac{1}{\cos 2\theta}$

$\therefore\quad f\left(\dfrac{1}{3}\right)=1-\sqrt{\dfrac{3}{2}}$ or $1+\sqrt{\dfrac{3}{2}}$

7. a., b., c.

$f(x)=(\log(\sec x+\tan x))^3\ \forall\ x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

$\therefore\quad f(-x)=(\log(\sec x-\tan x))^3$

$=\left(\log\dfrac{1}{\sec x+\tan x}\right)^3$

$=(-\log(\sec x+\tan x))^3$

$=-(\log(\sec x+\tan x))^3$

$=-f(x)$

Hence, $f(x)$ is odd function.

Let $g(x)=\sec x+\tan x\ \forall\ x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

So, $g'(x)=\sec x(\sec x+\tan x)>0\ \forall\ x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

$\Rightarrow g(x)$ is one-one function

Hence, $(\log_e(g(x)))^3$ is also one-one function.

And $g(x)\in(0,\infty)\ \forall\ x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

$g(x)=\sec x+\tan x$

$=\dfrac{1+\sin x}{\cos x}$

$=\dfrac{1-\cos\left(\dfrac{\pi}{2}+x\right)}{\sin\left(\dfrac{\pi}{2}+x\right)}=\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

Now $x\in\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)\Rightarrow\dfrac{\pi}{4}+\dfrac{x}{2}\in\left(0,\dfrac{\pi}{2}\right)\Rightarrow\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\in(0,\infty)$

$\Rightarrow\quad \log(g(x))\in R$

Hence, $f(x)$ is an onto function.

8. a., b., c.

$f(x)=\sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)$

We know that $-1\le\sin x\le 1$

$\Rightarrow\quad -\dfrac{\pi}{2}\le\dfrac{\pi}{2}\sin x\le\dfrac{\pi}{2}$

$\Rightarrow\quad -1\le\sin\left(\dfrac{\pi}{2}\sin x\right)\le 1$

$\Rightarrow\quad -\dfrac{\pi}{6}\le\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\le\dfrac{\pi}{6}$

$\Rightarrow\quad -\dfrac{1}{2}\le\sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)\le\dfrac{1}{2}$

Now, $fog(x)=\sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)\right)$

$-1\le\sin\left(\dfrac{\pi}{2}\sin x\right)\le 1$

$\Rightarrow\quad \dfrac{-\pi}{2}\le\dfrac{\pi}{2}\left(\sin\left(\dfrac{\pi}{2}\sin x\right)\right)\le\dfrac{\pi}{2}$

$\Rightarrow\quad -1\le\sin\left(\dfrac{\pi}{2}\left(\sin\left(\dfrac{\pi}{2}\sin x\right)\right)\right)\le 1$

$\Rightarrow\quad \dfrac{-\pi}{6}\le\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\left(\sin\left(\dfrac{\pi}{2}\sin x\right)\right)\right)\le\dfrac{\pi}{6}$

$\Rightarrow\quad \dfrac{-1}{2}\le f(x)\le\dfrac{1}{2}$

Thus, range of fog is also $\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$.

Now, $\lim\limits_{x\to 0}\dfrac{\sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)}{\dfrac{\pi}{2}\sin x}$

$=\lim\limits_{x\to 0}\dfrac{2}{\pi}\dfrac{\sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)}{\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)}\times\dfrac{\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)}{\dfrac{\sin x}{x}\times x}$

$=\lim\limits_{x\to 0}\dfrac{2}{\pi}\times\dfrac{\pi}{6}\times\dfrac{\sin\left(\dfrac{\pi}{2}\sin x\right)}{\dfrac{\pi}{2}\sin x}\times\dfrac{\dfrac{\pi}{2}\sin x}{x}$

$=\dfrac{1}{3}\times\dfrac{\pi}{2}=\dfrac{\pi}{6}$

$$gof(x) \in \left[-\frac{\pi}{2}\sin\left(\frac{1}{2}\right), \frac{\pi}{2}\sin\left(\frac{1}{2}\right)\right]$$

$\Rightarrow \quad gof(x) \neq 1$

Matching Column Type

1. (a) – (p), (r), (s); (b) – (q), (s); (c) – (q), (s); (d) – (p), (r), (s).

We have $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6} = \dfrac{(x-5)(x-1)}{(x-2)(x-3)}$

(a) – (p), (r), (s).

If $-1 < x < 1$, then $f(x) = \dfrac{(-\text{ve})(-\text{ve})}{(-\text{ve})(-\text{ve})} = +\text{ve}$

$\therefore f(x) > 0$

Also, $f(x) - 1 = \dfrac{-x-1}{x^2 - 5x + 6} = -\dfrac{(x+1)}{(x-2)(x-3)}$

For $-1 < x < 1, f(x) - 1 = \dfrac{-(+\text{ve})}{(-\text{ve})(-\text{ve})} = -\text{ve}$

or $f(x) - 1 < 0$ or $f(x) < 1$

$\therefore 0 < f(x) < 1$

(b) – (q), (s).

If $1 < x < 2$, then $f(x) = \dfrac{(-\text{ve})(+\text{ve})}{(-\text{ve})(-\text{ve})} = -\text{ve}$

Therefore, $f(x) < 0$ and, so, $f(x) < 1$.

(c) – (q), (s).

If $3 < x < 5$, then

$$f(x) = \frac{(-\text{ve})(+\text{ve})}{(+\text{ve})(+\text{ve})} = -\text{ve}$$

Therefore, $f(x) < 0$ and, so, $f(x) < 1$.

(d) – (p), (r), (s).

For $x > 5, f(x) > 0$. Also,

$$f(x) - 1 = \frac{-(x+1)}{(x-2)(x-5)} < 0 \text{ for } x > 5$$

or $\quad f(x) < 1$,

$\therefore 0 < f(x) < 1$

Alternative Method:

We have $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6} = \dfrac{(x-5)(x-1)}{(x-2)(x-3)}$

Lets draw the graph of $y = f(x)$.
The domain of function is $R - \{2, 3\}$.
Since $f(0) = 5/6$, graph cuts the y-axis at $(0, 5/6)$.
when $y = 0$; $x = 1, 5$.
Hence, graph intersects x-axis at $(1, 0)$ and $(5, 0)$.

$$\lim_{x \to 2^-} \frac{(x-5)(x-1)}{(x-2)(x-3)} = -\infty; \ \lim_{x \to 2^+} \frac{(x-5)(x-1)}{(x-2)(x-3)} = \infty$$

$$\lim_{x \to 3^-} \frac{(x-5)(x-1)}{(x-2)(x-3)} = \infty; \ \lim_{x \to 3^+} \frac{(x-5)(x-1)}{(x-2)(x-3)} = -\infty$$

Also, $\displaystyle\lim_{x \to \pm\infty} \frac{(x-5)(x-1)}{(x-2)(x-3)} = 1$

$$f'(x) = \frac{(2x-6)(x^2-5x+6) - (2x-5)(x^2-6x+5)}{(x^2-5x+6)^2}$$

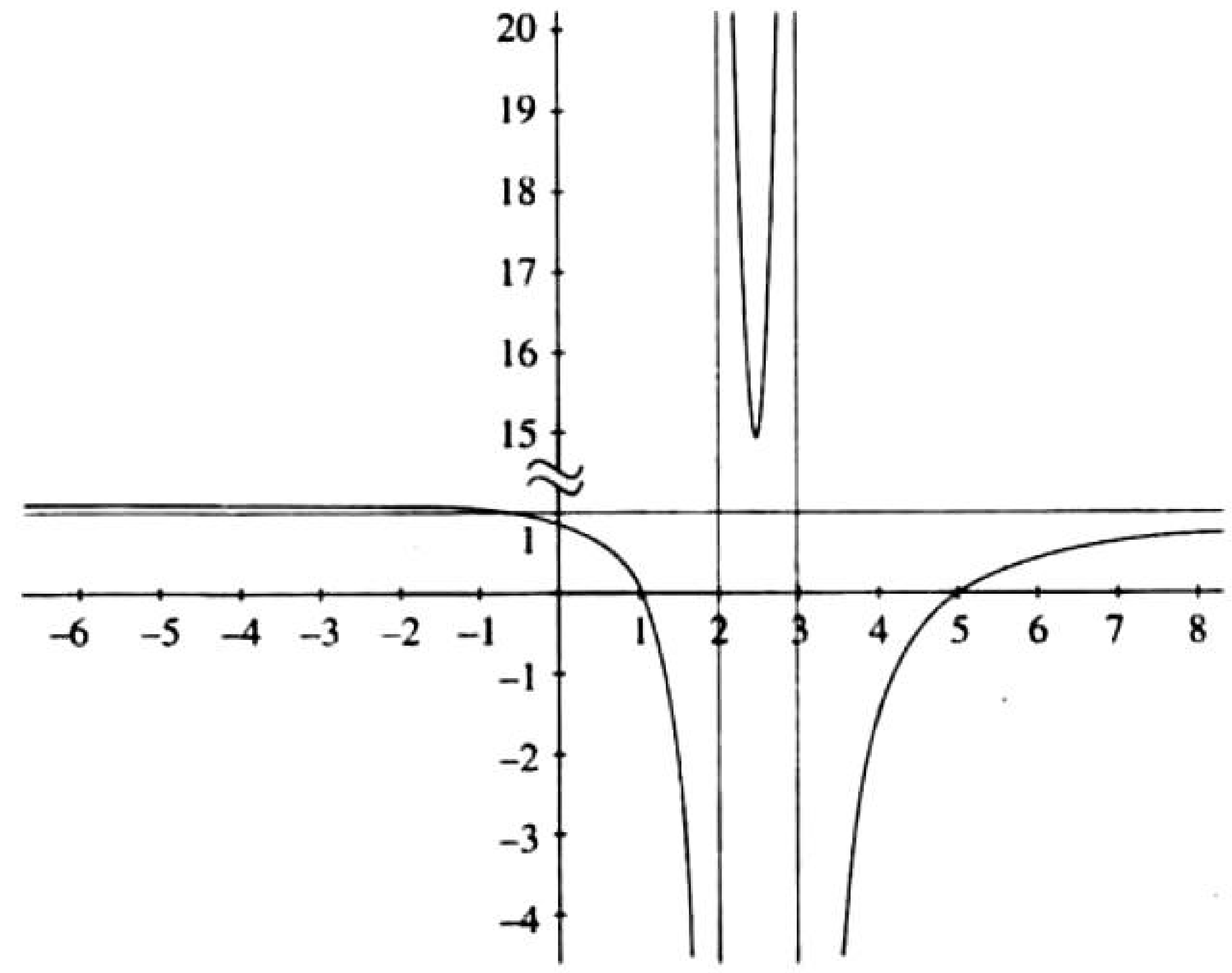
$$= \frac{x^2 + 2x - 11}{(x^2 - 5x + 6)^2}$$

$f'(x) = 0$

$\therefore \quad x = -1 \pm 2\sqrt{3}$

Clearly, $x = -1 + 2\sqrt{3}$ is point of minima and $x = -1 - 2\sqrt{3}$ is point of maxima.

From the above information, we can draw the graph of $y = f(x)$ as shown in the following figure.



From the graph for $-1 < x < 1, 0 < f(x) < 1$ (as $f(-1) = 1$)
For $1 < x < 2, f(x) < 0$
For $3 < x < 5, f(x) < 0$
For $x > 5, 0 < f(x) < 1$

2. (c) – (q), (r), (s), (t)

We have $f(x) = |x-1| + |x-2| + |x+1| + |x+2| = 4k$.

Clearly, for any integral value of x, $f(x)$ takes even integral value.

Also, least value of $f(x)$ is 6 for $-1 \le x \le 1$.

Thus the possible values of k are 2, 3, 4, 5.

Note: Solutions of the remaining parts are given in their respective chapters.

3. (b) – (t)

We know that $\sin^{-1} x$ is defined if $x \in [-1, 1]$.

So, given function $f(x) = \sin^{-1} \dfrac{8(3)^{x-2}}{1 - 3^{2(x-1)}}$ is defined if

$$-1 \le \frac{8.3^{x-2}}{1 - 3^{2(x-1)}} \le 1$$

or $\quad -1 \le \dfrac{3^x - 3^{x-2}}{1 - 3^{2x-2}} \le 1$

Now, $\dfrac{3^x - 3^{x-2}}{1 - 3^{2x-2}} - 1 \le 0$

$\therefore \quad \dfrac{(3^x - 1)(3^{x-2} + 1)}{(3^{2x-2} - 1)} \ge 0$

sign scheme:

$$\overset{+}{\quad}\ 0\ \overset{-}{\quad}\ 1\ \overset{+}{\quad}$$

$\therefore \quad x \in (-\infty, 0] \cup (1, \infty)$ (i)

Also, $\dfrac{3^x - 3^{x-2}}{1 - 3^{2x-2}} + 1 \geq 0$

$\therefore \dfrac{(3^{x-2} - 1)(3^x + 1)}{(3^x \cdot 3^{x-2} - 1)} \geq 0$

sign scheme:

$$\overset{+}{\quad}\ 1\ \overset{-}{\quad}\ 2\ \overset{+}{\quad}$$

or $x \in (-\infty, 1) \cup [2, \infty)$ (ii)

From (i) and (ii), $x \in (-\infty, 0] \cup [2, \infty)$

Note: Solutions of the remaining parts are given in their respective chapters.

Fill in the Blanks Type

1. For the given function to be defined,

$\dfrac{\pi^2}{16} - x^2 \geq 0$

or $-\pi/4 \leq x \leq \pi/4$

$\therefore D_f = [-\pi/4, \pi/4]$

Now, for $x \in [-\pi/4, \pi/4]$, $\sqrt{\pi^2/16 - x^2} \in [0, \pi/4]$ and sine function increases on $[0, \pi/4]$. Therefore,

$\sin 0 \leq \sin\sqrt{\dfrac{\pi^2}{16} - x^2} \leq \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$\therefore 0 \leq 3\sin\sqrt{\dfrac{\pi^2}{16} - x^2} \leq \dfrac{3}{\sqrt{2}}$

$\therefore f(x) \in \left[0, \dfrac{3}{\sqrt{2}}\right]$.

2. For $f(x)$ to be defined, we must have $-1 \leq \log_2\left(\dfrac{x^2}{2}\right) \leq 1$

or $2^{-1} \leq \dfrac{x^2}{2} \leq 2^1$ or $1 \leq x^2 \leq 4$

or $x \in [-2, -1] \cup [1, 2]$

3. Set A has n distinct elements

Then to define a function from A to A, we need to associate each element of set A to any one of the n elements of set A. So for each element we have 'n' options. Therefore number of functions is equal to total number of ways in which all elements are associated. The total number of such ways is $n \times n \times n \times \ldots \times n$ (n times) $= n^n$

Hence, the total number of functions form A to A is n^n.

Onto functions are those which have range equal to codomain, here set A. For $A \to A$, we need to associate each element of A to one and only one element of A. So total number of onto functions from set A to A is equal to number of ways of arranging n elements in range (set A) keeping n elements fixed in domain (set A). n elements can be arranged in n ! ways.

Hence, the total number of onto functions from A to A is $n!$.

4. The given function is, $f(x) = \sin\left[\ln\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)\right]$

Sine function is defined for all real numbers.

But logarithmic function is defined only for positive values.

Then $\dfrac{\sqrt{4-x^2}}{1-x} > 0$

$\Rightarrow \quad 1 - x > 0$ also $4 - x^2 > 0$

$\Rightarrow \quad x < 1$ and $-2 < x < 2$

$\Rightarrow \quad x \in (-2, 1)$

Now lets first find the values of $\ln\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$ for $x \in (-2, 1)$.

When $x \to 1^-$, $\dfrac{\sqrt{4-x^2}}{1-x}$ takes form $-\dfrac{\sqrt{3}}{0^+} \to \infty$

When $x \to -2^+$, $\dfrac{\sqrt{4-x^2}}{1-x} \to 0$

Thus $\dfrac{\sqrt{4-x^2}}{1-x} \in (0, \infty)$

$\Rightarrow \quad \ln\left(\dfrac{\sqrt{4-x^2}}{1-x}\right) \in (-\infty, \infty)$

$\Rightarrow \quad \sin\left(\ln\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)\right) \in [-1, 1]$

5. We have $f: [-1, 1] \to [0, 2]$, $f(x) = ax + b$.

Also, $f(x)$ is onto function.

So, range of function is $[0, 2]$ for domain $[-1, 1]$.

Therefore, $f(-1) = 0$ and $f(1) = 2$

or

$f(-1) = 2$ and $f(1) = 0$

Thus, there are two linear functions one passing through the points $(-1, 0)$ and $(1, 2)$ which is $f(x) = 1 + x$ and other passing through the points $(1, 0)$ and $(-1, 2)$ which is $f(x) = 1 - x$.

6. Given that $f(x) = f\left(\dfrac{x+1}{x+2}\right)$ and f is an even function

$\therefore f(x) = f(-x) = f\left(\dfrac{-x+1}{-x+2}\right)$

$\Rightarrow x = \dfrac{-x+1}{-x+2} \Rightarrow x^2 - 3x + 1 = 0$

$\Rightarrow x = \dfrac{3 \pm \sqrt{5}}{2}$.

Also $f(-x) = f\left(\dfrac{x+1}{x+2}\right)$

$\Rightarrow \dfrac{x+1}{x+2} = -x \Rightarrow x^2 + 3x + 1 = 0$

$\Rightarrow x = \dfrac{-3 \pm \sqrt{5}}{2}$.

$\therefore$ Four values of x are $\dfrac{3+\sqrt{5}}{2}, \dfrac{3-\sqrt{5}}{2}, \dfrac{-3+\sqrt{5}}{2}, \dfrac{-3-\sqrt{5}}{2}$.

7. $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$$= \sin^2 x + \frac{1}{4}\left(\sin x + \sqrt{3}\cos x\right)^2 + \frac{1}{2}\cos x\left(\cos x - \sqrt{3}\sin x\right)$$

$$= \frac{5}{4}\left(\sin^2 x + \cos^2 x\right) = \frac{5}{4}$$

$(gof)\,x = g[f(x)] = g(5/4) = 1$

True/False Type

1. $f(x) = (a - x^n)^{1/n}$, $a > 0$, n is +ve integer

or $f(f(x)) = f\left[\left(a - x^n\right)^{1/n}\right]$

$$= \left[a - \left\{\left(a - x^n\right)^{1/n}\right\}^n\right]^{1/n}$$

$$= \left(a - a + x^n\right)^{1/n} = x$$

Therefore, statement is true.

2. Since codomain is not given, we can assume codomain as R.

$$f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18} = \frac{(x+2)^2 + 26}{(x-4)^2 + 2} = y$$

For $y = 0$, there is no pre-image $x \in R$.

Therefore, f is not onto.

Hence, statement is true.

3. We know that the sum of any two functions is defined only on the points where both f_1 and f_2 are defined, that is, $f_1 + f_2$ is defined on $D_1 \cap D_2$.

Therefore, the given statement is false.

Subjective Type

1. Since $f(x)$ is defined and real for all real values of x, domain of f is R. Also,

$\frac{x^2}{1+x^2} \geq 0$ for all $x \in R$

and $\frac{x^2}{1+x^2} < 1 \quad (\because x^2 < 1 + x^2)$ for all $x \in R$

$\therefore\ 0 \leq \frac{x^2}{1+x^2} < 1 \quad$ or $\quad 0 \leq f(x) < 1$

Therefore, the range of $f = [0, 1)$.

Also, since $f(1) = f(-1) = 1/2$, f is not one-to-one.

Alternative Method:

We have $y = \frac{x^2}{1+x^2}$

$\Rightarrow \quad y + yx^2 = x^2$

$\Rightarrow \quad x^2 = \frac{y}{1-y}$

Now $x^2 \geq 0$

$\Rightarrow \quad \frac{y}{1-y} \geq 0$

$\Rightarrow \quad \frac{y}{y-1} \leq 0$

$\Rightarrow \quad 0 \leq y < 1$

Thus range is $[0, 1)$.

2. $y = |x|^{1/2}, -1 \leq x \leq 1$

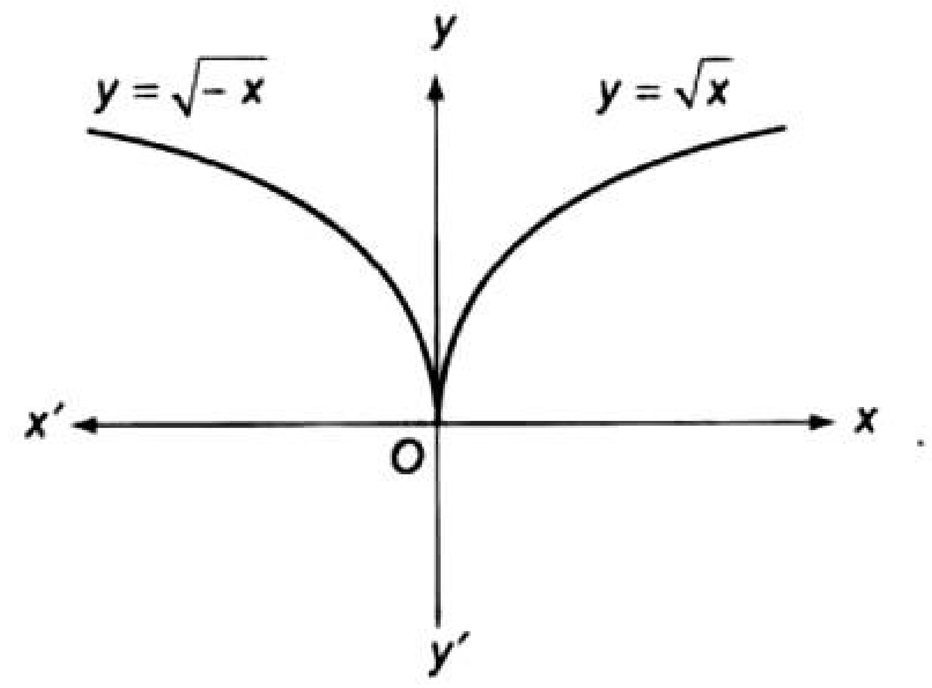
$$= \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$$

or $y^2 = -x$ if $-1 \leq x \leq 0$ and $y^2 = x$ if $0 \leq x \leq 1$

[Here, y should be always taken +ve, as by definition, y is a +ve square root].

Clearly, $y^2 = -x$ represents upper half of left-hand parabola (upper half as y is +ve) and $y^2 = x$ represents upper half of right-hand parabola.

Therefore, the resulting graph is shown as follows:



3. $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$

Then $f(6) = 6^9 - 6 \times 6^8 - 2 \times 6^7 + 12 \times 6^6 + 6^4 - 7 \times 6^3 + 6 \times 6^2 + 6 - 3$

$$= 6^9 - 6^9 - 2 \times 6^7 + 2 \times 6^7 + 6^4 - 7 \times 6^3 + 6^3 + 6 - 3$$

$$= -6^3 + 6^3 + 6 - 3$$

$$= 3$$

4. **Case I:** $f(x) \neq 2$ is true, $f(y) = 2$, and $f(z) \neq 1$ are false. Then $f(x) = 1$ or 3, $f(y) = 1$ or 3, and $f(z) = 1$.

Therefore, f is not one-one.

Case II: $f(x) \neq 2$ is false, $f(y) = 2$ is true, $f(z) \neq 1$ is false. Then $f(x) = 2$, $f(y) = 2$, $f(z) = 1$.

Therefore, it is not possible.

Case III: $f(x) \neq 2$ is false, $f(y) = 2$ is false, $f(z) \neq 1$ is true. Then $f(x) = 2$, $f(y) = 1$ or 3, $f(z) = 2$ or 3.

Thus, $f(x) = 2$, $f(z) = 3$, $f(y) = 1$.

5. Given that $f(x + y) = f(x)\,f(y)\ \forall\, x, y \in N$ and $f(1) = 2$

$f(2) = f(1 + 1) = f(1)\,f(1) = 2^2$

or $f(3) = f(2 + 1) = f(2)\,f(1) = 2^2 \times 2 = 2^3$

Similarly, $f(4) = 2^4, \ldots, f(n) = 2^n$

$$\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a) f(k)$$

$$= f(a)\sum_{k=1}^{n} f(k)$$

$$= f(a)[f(1) + f(2) + \cdots + f(n)]$$

$= f(a)[2 + 2^2 + \cdots + 2^n]$

$= f(a)\left[2\left(\dfrac{2^n - 1}{2 - 1}\right)\right]$

From $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, $f(a) = 8 = 2^3$ or $a = 3$.

6. Given that $4\{x\} = x + [x]$,
where $[x]$ = greatest integer $\le x$ and $\{x\}$ = fractional part of x.
We know that $x = [x] + \{x\}$, for any $x \in R$.
Therefore, given equation becomes

$4\{x\} = [x] + \{x\} + [x]$

or $3\{x\} = 2[x]$

or $[x] = \dfrac{3}{2}\{x\}$ (1)

Now, $0 \le \{x\} < 1$

or $0 \le \dfrac{3}{2}\{x\} < \dfrac{3}{2}$

or $0 \le [x] < \dfrac{3}{2}$ [Using (1)]

or $[x] = 0, 1$

If $[x] = 0$, then

$\dfrac{3}{2}\{x\} = 0$ [Using (1)]

or $\{x\} = 0$

$\therefore x = 0 + 0 = 0$

If $[x] = 1$, then

$\dfrac{3}{2}\{x\} = 1$ [Using (1)]

or $\{x\} = 2/3$

$\therefore x = 1 + 2/3 = 5/3$

Thus, $x = 0, 5/3$.

Alternative Method:

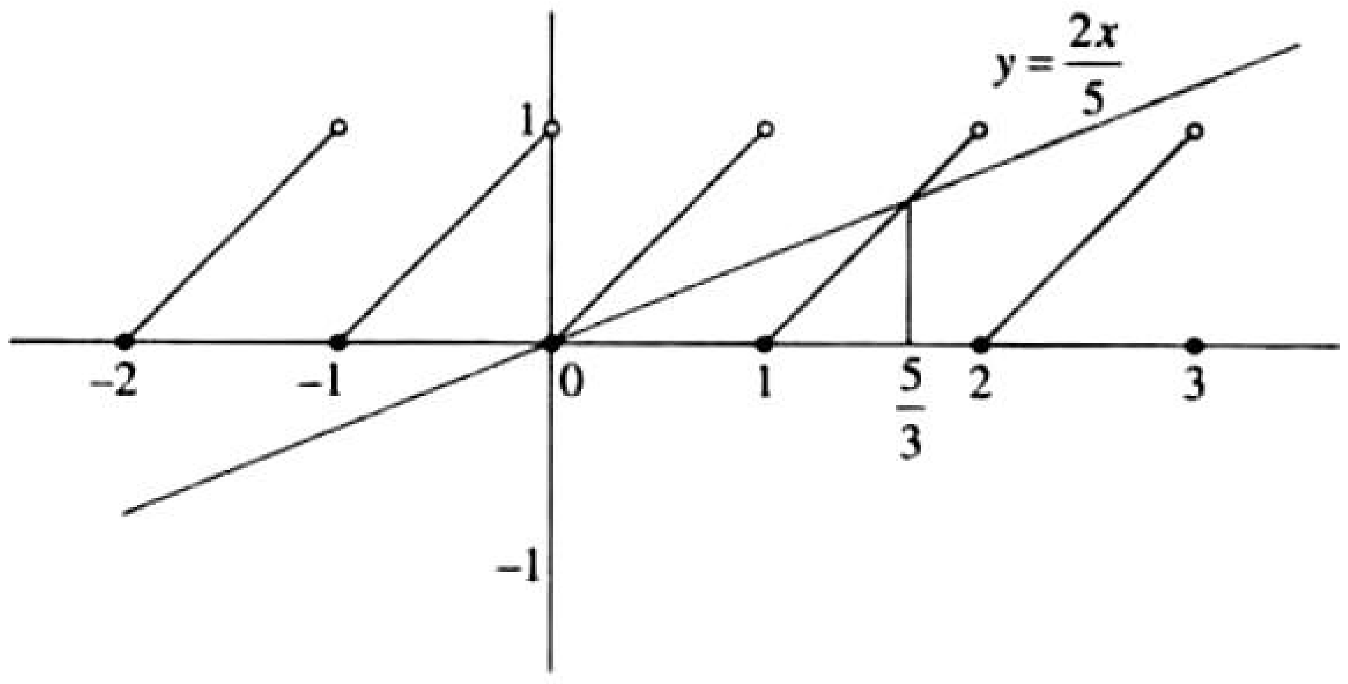
We have $4\{x\} = x + [x]$

or $4\{x\} = x + x - \{x\}$

or $5\{x\} = 2x$

or $\{x\} = \dfrac{2x}{5}$

The graphs of $y = \{x\}$ and $y = \dfrac{2}{5}x$ are as shown in the following figure.



From the graph, one of the solutions is $x = 0$.

Other solution occurs where $\dfrac{2}{5}x = x - 1$ or $x = \dfrac{5}{3}$.

7. Let us put $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

or $(\alpha + 6x - 8x^2)\, y = \alpha x^2 + 6x - 8$

or $(\alpha + 8y)x^2 + 6(1 - y)x - (8 + \alpha y) = 0$

Since x is real,

$36(1 - y)^2 + 4(\alpha + 8y)(8 + \alpha y) \ge 0$

or $9(1 - y)^2 + (\alpha + 8y)(8 + \alpha y) \ge 0$

or $y^2(9 + 8\alpha) + y(46 + \alpha^2) + (9 + 8\alpha) \ge 0$ (1)

Since function is onto, its range is R. So, (1) is satisfied for all real values of x.

For (1) to hold for each $y \in R$,

$\therefore 9 + 8\alpha > 0$ and $(46 + \alpha^2)^2 - 4(9 + 8\alpha)^2 \le 0$

$\therefore \alpha > -\dfrac{9}{8}$ and $[46 + \alpha^2 - 2(9 + 8\alpha)]\,[46 + \alpha^2 + 2(9 + 8\alpha)] \le 0$

$\therefore \alpha > -\dfrac{9}{8}$ and $(\alpha^2 - 16\alpha + 28)(\alpha^2 + 16\alpha + 64) \le 0$

$\therefore \alpha > -\dfrac{9}{8}$ and $(\alpha - 2)(\alpha - 14)(\alpha + 8)^2 \le 0$

$[\because (\alpha + 8)^2 \ge 0]$

$\therefore \alpha > -\dfrac{9}{8}$ and $2 \le \alpha \le 14$

$\therefore 2 \le \alpha \le 14$

Thus, $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ will be onto if $2 \le \alpha \le 14$.

When $\alpha = 3$, $f(x) = \dfrac{3x^2 + 6x - 8}{3 + 6x - 8x^2}$. In this case, $f(x) = 0$ implies $3x^2 + 6x - 8 = 0$

or $x = \dfrac{-6 \pm \sqrt{36 + 96}}{6} = \dfrac{-6 \pm \sqrt{132}}{6} = \dfrac{-6 \pm 2\sqrt{33}}{6}$

$= \dfrac{1}{3}\left(-3 \pm \sqrt{33}\right)$

Thus, $f\left[\dfrac{1}{3}\left(-3 + \sqrt{33}\right)\right] = f\left[\dfrac{1}{3}\left(-3 - \sqrt{33}\right)\right] = 0$.

Therefore, f is not one-to-one.